Phenomenological Three-Cluster Model of ^{6}He[*]

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Abstract

By using the method of hyperspherical functions within the appropriate for this method $K_{\min}$ approximation, the simple three-cluster model for description of the ground state and the continuous spectrum states of ^{6}He is developed. It is shown that many properties of ^{6}He (its large rms radius and large values of the matrix elements of electromagnetic transitions from the ground state into the continuous spectrum) follow from the fact that the potential energy of ^{6}He system decreases very slowly (as ρ^{-3}) and the binding energy is small.

1 Introduction

The ^{6}He nucleus is an example of a three-cluster system whose lowest threshold ($\alpha+n+n$) is a three-particle one. Such systems have a number of remarkable properties determined mainly by two factors—the Pauli principle and the character of the potential energy dependence due to the three-body structure.

The Pauli principle imposes essential restrictions on the wave function of a system allowing it to contain only the components antisymmetric with respect to nucleon permutation. Thus, if we are using the expansion over the harmonic-oscillator basis, we must solve the problem of excluding the states forbidden by the Pauli principle [1]. This, in particular, leads to the fact that the simplest wave function of the 0^+-state of ^{6}He obtained with the translation-invariant shell model (it coincides with the lowest oscillator-basis function) is a superposition of states with three-particle hypermomentum $K = 0$ ($\sim 5\%$) and $K = 2$ ($\sim 95\%$). This function generates an infinite set of states differing by a number of oscillator

quanta and having almost the same ratio of $K = 0$ and $K = 2$ components. The approximation using only this set of functions can be appropriately called the $K_{\min}$-approximation. Other basis states contain components with $K > 2$ whose weight is small, as shown in [2].

Notice that for 2^+-states, wave functions of the $K_{\min}$-approximation contain only $K = 2$ components, while for the 1^--states they are the superpositions of $K = 1$ ($\gtrsim 25\%$) and $K = 3$ ($\lesssim 75\%$) components.

The second characteristic feature of systems under consideration is the slow decrease of their potential energy for large values of the hyperradius ρ. As known (see for example [3]), in three-body systems, the potential energy has the asymptotical behaviour const$/\rho^3$ as $\rho \to \infty$, even if the two-body forces acting between any two of constituent particles are short-range. This fact leads to two important results. First, the boundary of a system is extremely loose and the rms radius is significantly larger compared with that of those neighbouring nuclei whose three-particle threshold is much higher than the binary one. Second, the phase shift of the three-body scattering rises sharply at low energies (it is proportional to $\sqrt{E}$ as $E \to 0$) which directly affects the behaviour of the electromagnetic transitions matrix elements between the ground state of ^{6}He and its continuous spectrum.

The aim of the present paper is to give a detailed quantitative analysis of general regularities in ^{6}He structure, based on a simple model taking into account the main features of a loosely-bound system with the lowest three-particle threshold.

2 The Model

The allowed basis functions for states with zero angular momentum $\varphi_n(L = 0)$ of the $K_{\min}$-approximation have the following form

$$\varphi_n(L = 0) = A_0(n)\,\varphi_{n0}(\rho)\,u_0(\omega) + A_2(n)\,\varphi_{n2}(\rho)\,u_2(\omega), \tag{1}$$

where the superposition coefficients are

$$A_0(n) = \sqrt{\frac{4(n+1)}{29n + 104}}, \quad A_2(n) = \sqrt{\frac{25(n+4)}{29n + 104}}, \tag{2}$$

$u_K(\omega)$ are the hyperspherical functions,

$$\varphi_{nK}(\rho) = N_{nk}\rho^K e^{-\rho^2/2} L_n^{K+2}(\rho^2) \tag{3}$$

are the normalized hyperradial harmonic-oscillator basis functions, $L_n^\alpha(\rho^2)$ are the Laguerre polynomials, N_{nk} are the normalizing coefficients, $n = 0, 1, \ldots$ Therefore, the wave function of the $L = 0$ state in the $K_{\min}$-approximation is reduced to the expansion

$$\Psi(L = 0) = \sum_{n=0}^{\infty} C_n \varphi_n(L = 0). \tag{4}$$

For all n, the contribution of $K = 2$ states is significantly greater than that of the $K = 0$ states

$$0.86 < A_2^2(n) < 0.96; \quad A_0^2(n) = 1 - A_2^2(n),$$

hence, in a simple model, the contribution of $K = 0$ states can be neglected. Then, for the expansion (4) the following relations hold

$$\Psi(L=0) = \sum_{n=0}^{\infty} C_n \varphi_{n2}(\rho)\, u_2(\rho) = \Phi(\rho)\, u_2(\Omega) = \frac{\chi(\rho)}{\rho^{5/2}} u_2(\Omega). \tag{5}$$

The function $\chi(\rho)$ should satisfy the one-dimensional Schrödinger equation

$$\frac{\hbar^2}{2m}\left[-\frac{d^2\chi}{d\rho^2} + \left(K+\frac{3}{2}\right)\left(K+\frac{5}{2}\right)\frac{\chi}{\rho^2}\right] + V(\rho)\,\chi = E\chi, \tag{6}$$

where m is the nucleon mass, K is the hypermomentum ($K = 2$ for $L = 0$), $V(\rho)$ is the effective three-body potential.

The equation (6) is a starting point for the subsequent discussion. It was solved by a numerical integration over ρ from zero to a certain sufficiently large cutoff radius $\rho_{\max}$. The effective potential is modelled by a function

$$V(\rho) = \frac{V_0}{1+\left(\dfrac{\rho}{a}\right)^3}, \tag{7}$$

having correct asymptotics and without a singularity at $\rho = 0$. To reproduce the states with zero angular momentum, we have chosen the parameters V_0 and a such that the equation (6) would give experimental values for the ^{6}He binding energy and the rms radius, $E = -\mathcal{E} = -0.97$ MeV, $\sqrt{\langle r^2\rangle} = 2.57$ Fm. The appropriate values of the parameters are

$$V_0 = -87 \text{ MeV}, \quad a = 3.073 \text{ Fm}. \tag{8}$$

With these values we have calculated the wave function of the ^{6}He ground state and 0^+-states of its continuous spectrum.

3 The Ground State of ^{6}He

The wave function of the ground state of ^{6}He together with the effective potential $V(\rho)$ are presented in Fig. 1. The horizontal line below the ρ-axis corresponds to the ground state energy, its intersection with the potential energy curve marks the classical turning point. The vertical line separates the values of ρ less than $\sqrt{\langle\rho^2\rangle} = 5.59$ Fm. The wave function falls as $\rho^{K+5/2}$ for small ρ, due to the strong kinematical barrier, while for large ρ, the long-range character of the potential $V(\rho)$ leads to the slow decrease of the wave function and, respectively, to a significant diffuseness of the boundary of a nuclear system.

As known, loosely-bound binary systems with a short-range potential can be rather well approximated outside the potential range by the exponential function

$$\chi(\rho) \simeq \sqrt{2\kappa}\exp(-\kappa r), \qquad \kappa = \sqrt{\frac{2m}{\hbar^2}\mathcal{E}}\,, \tag{9}$$

where $\mathcal{E}$ is the bound state energy, r is the radial variable of the binary channel. The formal criterion of validity of such an approximation (the zero-range approximation) is the smallness of a ratio of the potential range and the cluster radii to the system radius $1/\sqrt{2\kappa}$ expressed

in terms of binding energy. It seems useful to compare the exact wave function obtained after the solution of Eq. (6) with the approximation given by Eq. (9) (for $\kappa \approx 0.22$ Fm^{-1}, dashed line in Fig. 1). It can be seen that the nucleon system is much more loose than it is predicted by the wave function (9). The reason for that is not only that the function $\chi(\rho)$ at small ρ behaves as $\rho^{K+5/2}$ as $\rho \to 0$, but first of all that the effective potential energy decreases slower than the exponential, while the formula (9) is obtained by supposing that the potential is short-range. As a result, the estimates of the square of hyperradius mean value $\sqrt{\langle\rho^2\rangle}$ based on (9) are about three times less than the experimental value.

Of course, for sufficiently large ρ the function $\chi(\rho)$ has the asymptotics

$$\chi(\rho) \simeq C\sqrt{\kappa\rho}\, K_4(\kappa\rho) \simeq C \exp(-\kappa\rho), \tag{10}$$

where $K_4(\rho)$ is the Macdonald function, and the value of the coefficient C (it is related to the so-called nuclear vertex constants, see [4]) is considerably less than that of $\sqrt{2\kappa}$

$$C \approx 0.12 \text{ Fm}^{-1/2} < \sqrt{2\kappa} \approx 0.66 \text{ Fm}^{-1/2}. \tag{11}$$

This asymptotics is denoted by short-dashed line in Fig. 1.

4 0^+-States of the Continuous Spectrum

Of the special interest are the solutions of Eq. (6) for the continuous spectrum for relatively low over-threshold energies. In particular, attention must be paid to the question of the behaviour of the three-to-three scattering phase δ as a function of energy in a potential field with the asymptotical behaviour const/ρ^3 and the powerful kinematic barrier $(\hbar^2/2m)(63/4\rho^2)$ corresponding to $K = 2$. According to the estimates [5, 6], at low energies, the phase shift, as is also demonstrated by our calculations, is proportional to k (or $\sqrt{E}$)

$$\delta \simeq Ak + \ldots\,; \qquad A = 3.95 \text{ Fm}. \tag{12}$$

The low-energy values of $\tan\delta$ calculated with different values of the cutoff radius $\rho_{\max}$ are presented as functions of k in Fig. 2. As seen, $\rho_{\max}$ should be rather large, at least 1000 Fm for energies below 10 keV. The obtained value of A gives a rather reasonable prediction for the ground-state energy of ^{6}He

$$\mathcal{E}_{\text{approx.}} = \frac{\hbar^2}{2m}\frac{1}{A^2} \approx 1.3 \text{ MeV}, \quad (\text{cf. } \mathcal{E}_{\text{exp.}} = 0.97 \text{ MeV}). \tag{13}$$

The three-body phase shift δ calculated with different $\rho_{\max}$ as functions of energy E in an interval up to 5 MeV are presented in Fig. 3. As seen, for reliable calculations for medium-energy region (~ 1 MeV) $\rho_{\max}$ may be taken about 50 Fm and only for the lowest energies it must be increased. The phase shift rises steeply from zero to values exceeding 90° for $E \approx 2.5$ MeV and then slowly goes down. When the phase shift is near 90° the first maximum of the wave function moves closer to zero so that the matrix element of the isoscalar transition from the ground state into the continuous spectrum increases (see Fig. 4). This matrix element reaches its maximal value for $E \approx 1.3$ MeV which could directly affect the electrodisintegration cross-section of ^{6}He.

5 2^+-States of the Continuous Spectrum

The K_{min}-approximation satisfying the Pauli principle for the 2^+-states of ^{6}He contain only the hyperspherical function with $K = 2$. Therefore, to obtain the wave functions $\chi(\rho)$ of 2^+-states, we again turn to the equation (6) with $K = 2$ but with the other parameters of the potential $V(\rho)$.

The ^{6}He nucleus has no 2^+ bound state but it has a 2^+-resonance at the energy of 0.822±0.025 MeV with the width 0.113±0.020 MeV [7]. The energy and the width of the resonance can be reproduced with the following parameters of potential:

$$V_0 = -92 \text{ MeV}, \quad a = 2.834 \text{ Fm}. \tag{14}$$

The wave function $\chi(\rho)$ of the resonance state and the potential energy with the parameters (14) are presented in Fig. 5. For small ρ the resonance wave function behaves similarly to the wave function of the ground state (dashed line in Fig. 5) but for larger ρ, in the asymptotical region it, as it should be, oscillates.

The scattering phase shift in the 2^+-state obtained by solving the equation (6) with the new potential parameters is presented in Fig. 6. At the energy of about 1 MeV the phase shift has a typical behaviour for a resonance region.

We have also calculated the matrix element of the operator of the isoscalar $E2$ transition from the ground state of ^{6}He to the 2^+-states of its continuous spectrum. The dependence of this matrix element upon the energy is presented in Fig. 7. The narrow peak observed for the energy about 0.8 MeV again demonstrates the small width of 2^+ resonance and the large value of the matrix element.

6 Conclusion

The simple three-cluster model based on the phenomenological long-range potential with the Pauli principle taken into account has allowed us to reveal a number of regularities both for the weakly-coupled ground state of ^{6}He and for the states of its continuous spectrum for the relatively low (up to a few MeV) energies. The large value of the rms radius in the ground state can be explained by the considerable diffuseness of the wave function caused by the slowly-decreasing effective potential. The asymptotic region, where the wave function decreases exponentially begins only at the hyperradius values greater than 15–20 Fm.

The phase shift of elastic scattering ($3 \to 3$) is proportional to k or $\sqrt{E}$ at low energies which leads to the sharp maximum of the matrix element of the isoscalar transition from the ground state to the 0^+-states of the continuous spectrum. Finally, our calculations predict a considerable enhancing of the probability of the radiative capture of two neutrons by the alpha-particle for the energy corresponding to the 2^+-resonance of ^{6}He.

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References

[1] G. F. Filippov, A. V. Nesterov, I. Yu. Rybkin, S. V. Korennov, Fiz. Elem. Chastits. At. Yadra **25**, 1347 (1994).

[2] M. V. Zhukov, B. V. Danilin, D. V. Fedorov *et al.*, Phys. Rep. **231**, 151 (1993).

[3] A. M. Badalyan, Yu. A. Simonov, Yad. Fiz. **3**, 1032 (1066).

[4] L. D. Blokhintsev, I. Borbély, E. I. Dolinsky, Yad. Fiz. **8**, 1189 (1977).

[5] B. R. Levy, J. B. Keller, J. Math. Phys. **4**, 54 (1963).

[6] L. D. Landau, E. M. Lifshitz, *Quantum Mechanics* (Nauka, Moscow, 1989)

[7] F. Ajzenberg-Selov, Nucl. Phys. **A490**, 1 (1988).

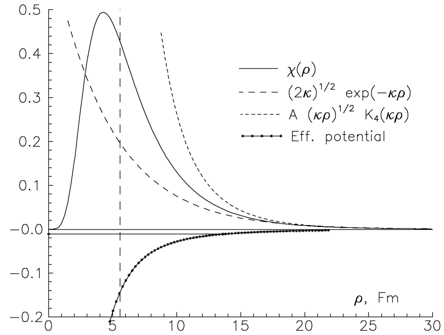

Figure 1: The wave function of ^{6}He ground state, its asymptotic behaviour (Eqs. (9) and (10)) and the effective potential. The ground-state energy is marked by a horizontal line, the vertical line corresponds to $\rho = \sqrt{\langle \rho^2 \rangle} = 5.59$ Fm).

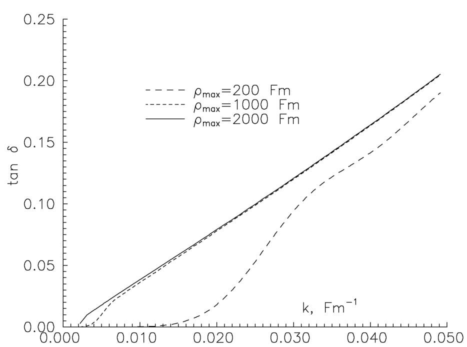

Figure 2: Low-energy behaviour of $\tan\delta$ as a function of k.

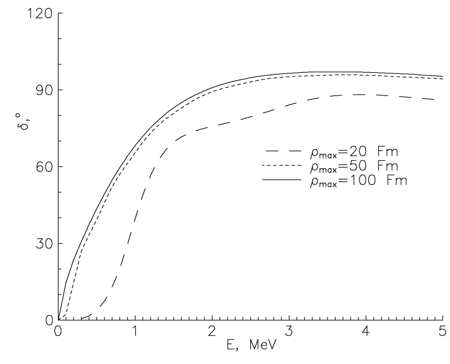

Figure 3: The scattering phase shift in the 0^+-state.

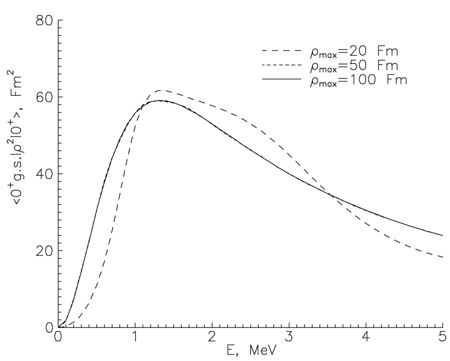

Figure 4: The matrix element of the isoscalar transition from the ground state to the 0^+ states of the continuous spectrum. (Results for $\rho_{\max} = 50$ Fm and 100 Fm are almost identical.)

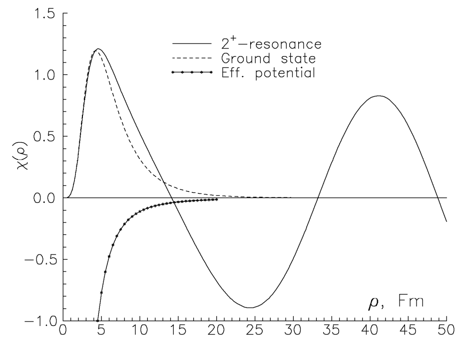

Figure 5: The wave function of the 2^+ resonance and the ground state (scaled) and the effective potential in the 2^+ state

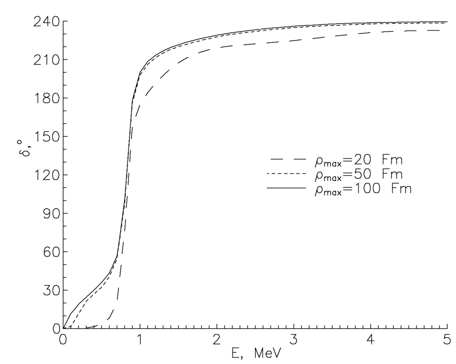

Figure 6: The scattering phase shift in the 2^+-state.

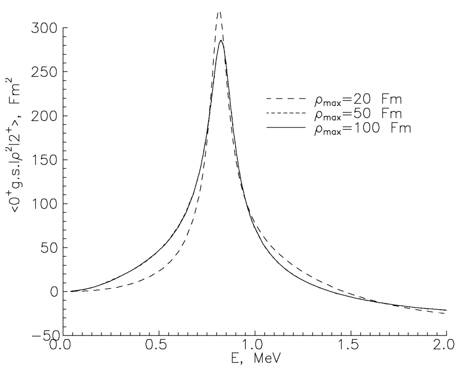

Figure 7: The matrix element of the isoscalar $E2$ transition from the ground state to the 2^+ states of the continuous spectrum. (Results for $\rho_{\max} = 50$ Fm and 100 Fm are almost identical.)